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DETERMINATION AND ANALYSIS OF NUMERICAL SMOOTHING WEIGHTS

by Ronald J. Graham

George C. Marshall Space Flight Center

Huntsville, Ala.

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DEFINITION OF TERMS AND SYMBOLS

SYMBOL	DEFINITION
Frequency	Cycles per time period, usually cycles per second.
Amplitude	The displacement of a function from its zero value.
(f)	The variable used to represent frequencies occurring in data in the time domain.
(ω)	The angular frequency, generally used in the computations in the frequency domain ($\omega = 2\pi f$). $\pi = 3.1415926...$
f_c	The largest frequency of a low-pass filter whose amplitudes are passed with unity gain (cycles per second).
f_t	The first frequency of a low-pass filter whose amplitudes are given a zero gain (cycles per second).
f_s	The sampling frequency (samples per second)
Δt	The time interval between samples or the sampling period, i.e., $\Delta t = \frac{1}{f_s}$.
Filter	An object or function operating on frequencies, referred to also as a gain function.
Gain function	The function in the frequency domain that states what coefficient will be a multiplier of each frequency's amplitude.
Frequency domain	Used here as the geometrical plane in which frequency is taken as the independent variable and gain functions of frequency as the dependent variable.
Time domain	The geometrical plane in which time is the independent variable and functions of time the dependent variable.
Weight function	The function of time resulting from the transformation of a filter or gain function from the frequency domain to the time domain.
Weight	A numerical value obtained from the weight function to be applied to the data in the time domain.

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SUMMARY

The purpose of this report is to present a systematic approach to the derivation and analysis of numerical smoothing weights. The text goes into a detailed step-by-step analysis of the method used and an appendix illustrates some actual results.

In almost all analyses of test data results, smoothing must be applied to numerical data. Almost all numerical analysts are aware of several different methods of smoothing. When the question is asked of what method is best, there is no direct, simple answer. Before attempting to answer this question, the data characteristics must be known.

Suppose significant characteristics of the data are known - the answer is still not unanimous. Numerical smoothing methods lack clear definitions. In general, there is no standard of comparison between different numerical smoothing techniques.

It is possible to compare almost all numerical smoothing procedures on the same basis by an extension of the method used in this paper. By using the method discussed, numerical smoothing formulae can be derived with specified characteristics for specific applications. Formulae that smooth and differentiate simultaneously can be derived by extending the concept discussed. Other related formulae can also be derived.

The known characteristics of the data are assumed to be the frequency characteristics of the data. Many times the significant frequency characteristics are known or can be determined. In cases where they are not known the validity of applying any smoothing formula may be questioned. (A study and report of methods to determine frequency characteristics of data is being planned.)

SECTION I. DISCUSSION

Often in electronic terminology, certain undesirable signals or indications are referred to as high-frequency noise. A 60-cycle per second interference may also be referred to as noise. Filters are used to eliminate known undesirable noise and to "clean" the data. Filters are selected such that frequencies carrying valid data of specific interest are allowed to pass with a specified gain function while other frequencies are rejected. The desired gain function is basic in determining the filter design. Once the filter is designed the response or output characteristics must be known or evaluated. Most desired gain functions can be represented by a mathematical function definition in the frequency domain for all frequencies. This is equal to the desired filter. However, the output of the desired filter is not completely precise. This is demonstrated by the Gibbs Phenomenon. An accepted technique is to allow tolerance in the filter design, thus yielding more precise output results. The design, development, and use of electronic filters is a scientific part of electronic engineering.

Many numerical smoothing formulae used by digital data processing experts are inherited ("hand-me-downs") or taken from famous textbooks. In some cases the applications of these methods must be considered unscientific. However, numerical smoothing can be accomplished on the same scientific basis as electronic filtering by using the method discussed in this text. Numerical methods have advantages over electronic methods. Some of these advantages will be listed later.

The filter or gain function is defined in the frequency domain; that is, where frequency (ω) is the independent variable and gains of frequencies ($H(\omega)$) is the dependent variable. Data are finally analyzed in the time domain, where variables can be expressed as functions of time.

The gain function or filter in the frequency domain can be transformed to a weight function in the time domain and vice-versa. The first function is an inverse Fourier transformation. The second function is a direct Fourier transformation.

Since these transformations can be applied, a smoothing function in the time domain can be transformed to a filter in the frequency domain, and a filter in the frequency domain can be transformed into a smoothing function in the time domain.

In electronics, filters and filtering techniques are widely used. Filters are usually specific parts of electronic hardware. An analog electronic computer performs the operations of smoothing, differentiating, and integrating in the time domain. These operations are analogous to filtering in the frequency domain.

Rough or unsmoothed numerical data can also be considered as data containing noise. Some erroneous or undesirable data, similar to high-frequency noise, are recognized by a sudden change in magnitude within a small time interval. Numerical data may also contain a slow varying bias which may be considered low-frequency noise. Numerical data, such as round-off error, may contain cyclic error that could be considered constant-frequency noise. Smoothing can be used to eliminate these types of undesirable noise. As in electronic data, certain numerical digital data may contain composite data values. By selecting proper weighting formulae these composite data values can sometimes be decomposed.

Numerical smoothing weights can be derived by identifying frequencies of interest or by identifying the undesirable frequencies. When these frequencies are determined, the gain function must be defined for all frequencies. This gain function defined for all frequencies is the filter. The value of the gain function at the frequencies that are to be eliminated or not considered should be zero. The value of the gain function where the frequencies are to pass through undisturbed should be one. An inverse Fourier transformation should be applied to this gain function. The result will be a weight function in the time domain.

The problem of determining smoothing weights in the time domain, therefore, is transformed into a problem in designing a filter in the frequency domain. Some advantages of designing a filter to determine smoothing weights rather than using other techniques in the time domain follow:

1. Filters are easily designed.

2. The design of filters has great flexibility.

3. The method is systematic and has a straight-forward approach.

4. Filters can be used as a standard to show the extent of smoothing.

5. The output of filters can be evaluated before application.

An infinite number of types or sets of filters can be designed. A simple, logical type of filter for determining smoothing weights is given in this report.

The technique of designing a filter and determining related numerical weights in the time domain is called numerical filtering. Some advantages of doing numerical filtering rather than electronic filtering follow:

1. Numerical filtering has higher fidelity. (Digital calculations can be repeated identically.)

2. The problem of phase shift can be diminished.

3. Feedback problems can be eliminated.

4. Numerical filters are not hardware bound and can be designed readily.

5. Numerical filters can be changed instantaneously.

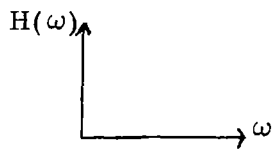
6. Numerical filtering can be used more directly in high-speed digital computer applications.

7. Numerical filtering can be done after electronic filtering to obtain more precision, and to extract erroneous signals such as feedback.

One of the drawbacks of numerical methods such as this one has been the requirement of many digital calculations. However, with the great advancement in high-speed electronic digital computers, such numerical methods are becoming more feasible.

SECTION II. THE FILTER DESIGN

To design a filter, work is done in the frequency domain.



ω, the frequency, is the independent variable. $H(\omega)$ is the gain function of the frequencies $(-\infty$ to $+\infty)$. This function is necessary to allow application of the Inverse Fourier Transformation. Of the infinite possible designs, a simple one is chosen for this report. This design is a low-pass filter with the following functional definition:

$$H(\omega) = H(-\omega)$$

$$H(\omega) = 0; \ |\omega| \geq \omega_t \quad \text{(the termination frequency)}$$

$$H(\omega) = 1; \ |\omega| \leq \omega_c \quad \text{(the cutoff frequency)}$$

$$H(\omega) = \frac{1}{2}\left\{\cos\left[\left(\frac{\omega_c + \omega}{\Delta\omega}\right)\pi\right] + 1\right\}; \ -\omega_t \leqq \omega \leqq -\omega_c$$

$$H(\omega) = \frac{1}{2}\left\{\cos\left[\left(\frac{\omega - \omega_c}{\Delta\omega}\right)\pi\right] + 1\right\}; \ \omega_c \leqq \omega \leqq \omega_t$$

$$\Delta\omega = \omega_t - \omega_c$$

From this basic low-pass filter design, the following type filters can be derived:

1. High-pass filter

2. Band-pass filter

3. Band-reject or notch filter

4. A linear combination of these three and low-pass filter.

The high-pass filter is the complement of the low-pass filter, i.e., their sum yields a filter that allows all frequencies to pass. The all-pass filter allows all frequencies to pass as they are. It has weights of all zero, except that the central weight is unity, that is, 0, 0, . . , 1, . . , 0, 0. The weights for the high-pass filter can be obtained by subtracting the low-pass weights from the all-pass weights.

The band-pass filter is the difference between two low-pass filters. The weights of the band-pass filter can be obtained by taking corresponding weight differences between two low-pass filters.

The band-reject or notch-filter weights can be obtained by subtracting the band-pass weights from the all-pass weights.

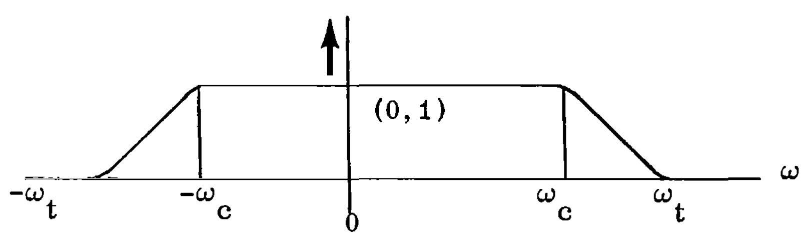
Linear combinations of effective filters can be obtained by taking linear combinations of corresponding weights.

Results to substantiate these statements are given in the appendix.

The inverse Fourier transformation is applied to the basic low-pass filter. Other weights can be derived as explained above.

SECTION III. THE INVERSE FOURIER TRANSFORMATION

The mathematical transformation is presented here in detail. Equation 2 is the resulting weight equation. $|H(\omega)| = H(\omega)$



Function definition:

$$H(\omega) = \begin{bmatrix} 0 & ; \; |\omega| > \omega_t \\ 1 & ; \; |\omega| \leqq \omega_c \\ \dfrac{\cos\left[\left(\dfrac{\omega_c + \omega}{\Delta\omega}\right)\pi\right] + 1}{2} & ; \; -\omega_t \leqq \omega \leqq -\omega_c \\ \dfrac{\cos\left[\left(\dfrac{\omega - \omega_c}{\Delta\omega}\right)\pi\right] + 1}{2} & ; \; \omega_c \leqq \omega \leqq \omega_t \end{bmatrix}$$

Find:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} H(\omega)\, d\omega \tag{1}$$

from

$$h(t) = A + B + C$$

where

$$A = \frac{1}{2\pi}\int_{-\omega_t}^{-\omega_c} \left\{ \frac{\cos\left[\left(\dfrac{\omega + \omega_c}{\Delta\omega}\right)\pi\right] + 1}{2} \right\} e^{i\omega t}\, d\omega$$

$$B = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{i\omega t}\, d\omega = \frac{\sin \omega_c t}{\pi t}$$

$$C = \frac{1}{2\pi}\int_{\omega_c}^{\omega_t} \left\{ \frac{\cos\left[\left(\dfrac{\omega - \omega_c}{\Delta\omega}\right)\pi\right] + 1}{2} \right\} e^{i\omega t}\, d\omega$$

and

$$\Delta\omega = \omega_t - \omega_c$$

where

$$A = A_1 + A_2$$

$$A_1 = \frac{1}{4\pi} \int_{-\omega_t}^{-\omega_c} \cos\left[\left(\frac{\omega + \omega_c}{\Delta\omega}\right)\pi\right] e^{i\omega t}\, d\omega$$

$$A_2 = \frac{1}{4\pi} \int_{-\omega_t}^{-\omega_c} e^{i\omega t}\, d\omega$$

also,

$$C = C_1 + C_2$$

with

$$C_1 = \frac{1}{4\pi} \int_{\omega_c}^{\omega_t} \cos\left[\left(\frac{\omega - \omega_c}{\Delta\omega}\right)\pi\right] e^{i\omega t}\, d\omega$$

$$C_2 = \frac{1}{4\pi} \int_{\omega_c}^{\omega_t} e^{i\omega t}\, d\omega$$

In solving A_1,

$$\text{set } x = \frac{\omega + \omega_c}{\Delta\omega},$$

then

$$\omega = \Delta\omega x - \omega_c, \quad \Delta\omega\, dx = d\omega.$$

When

$$\omega = -\omega_c, \quad x = 0,$$

and when

$$\omega = -\omega_t, \quad x = -1.$$

Hence,

$$A_1 = \frac{1}{4\pi} \int_{-1}^{0} \cos \pi x \, e^{i(\Delta\omega x - \omega_c)t} \, dx(\Delta\omega)$$

$$= \frac{\Delta\omega}{4\pi} e^{-i\omega_c t} \int_{-1}^{0} \cos \pi x \, e^{i\Delta\omega xt} \, dx = \frac{\Delta\omega}{4} e^{-i\omega_c t}$$

$$\left[\frac{e^{i\Delta\omega xt}}{\pi^2 - (\Delta\omega_t)^2} (i\Delta\omega t \cos \pi x + \pi \sin \pi x) \right]_{-1}^{0} = \frac{\Delta\omega}{4\pi} e^{-i\omega_c t}$$

Integration formula No. 314 Burington:

$$\left[\frac{i\Delta\omega t}{\pi^2 - (\Delta\omega t)^2} - \frac{e^{-i\Delta\omega t}}{\pi^2 - (\Delta\omega t)^2} (-i\,\Delta\omega t) \right] = \frac{\Delta\omega \; (\Delta\omega t)\, i}{4\pi\, [\pi^2 - (\Delta\omega t)^2]}$$

$$\left[e^{-i\omega_t t} + e^{-i\omega_t t} \right] = A_1^*.$$

$$A_2 + C_2 = \frac{1}{4\pi} \int_{-\omega_t}^{-\omega_c} e^{i\omega t} \, d\omega + \frac{1}{4\pi} \int_{\omega_c}^{\omega_t} e^{i\omega t} \, d\omega$$

$$= \frac{1}{4\pi it} \left[e^{i\omega t} \Big|_{-\omega_t}^{-\omega_c} + e^{i\omega t} \Big|_{\omega_c}^{\omega_t} \right] = \frac{1}{4\pi it} \; e^{-i\omega_c t} - e^{-i\omega_t t} + e^{i\omega_t t} - e^{i\omega_c t}$$

$$= \frac{1}{4\pi it} \; 2i \sin \omega_t t - 2i \sin \omega_c t = \frac{1}{2\pi t} \; \sin \omega_t t - \sin \omega_c t$$

$$= A_2^* + C_2^*$$

Solving for C_1,

$$\text{Set } y = \frac{\omega - \omega_c}{\Delta\omega}, \quad \Delta\omega \, dy = d\omega, \quad \omega = \Delta\omega y + \omega_c$$

then

$$C_1 \doteq \frac{1}{4\pi} \int_{0}^{1} \cos \pi y \; e^{i(\Delta\omega y + \omega_c)t} = \Delta\omega dy$$

when

$$\omega = \omega_t, \; y = 1$$

and when

$$\omega = \omega_c, \; y = 0$$

$$\frac{\Delta\omega e^{i\omega_c t}}{4\pi} \int_0^1 \cos \pi y \, e^{i\Delta\omega y t} \, dy = \frac{\Delta\omega e^{i\omega_c t}}{4\pi} \left. \frac{(e^{i\Delta\omega y t})}{(\pi^2 - (\Delta\omega t)^2} (i\Delta\omega t \cos \pi y + \pi \sin \pi y) \right]_0^1$$

$$= \frac{\Delta\omega e^{i\omega_c t}}{4\pi} \left[\frac{e^{i\Delta\omega t}(-i\Delta\omega t)}{\pi^2 - (\Delta\omega t)^2} - \frac{i\Delta\omega t}{\pi^2 - (\Delta\omega t)^2} \right]$$

$$= \frac{\Delta\omega(\Delta\omega t) i}{4\pi \; [\pi^2 - (\Delta\omega t)^2]} \left[-e^{i\omega_t t} - e^{i\omega_c t} \right] = C_1^*$$

$$A_1^* + C_2^* = \frac{\Delta\omega \; (\Delta\omega t) i}{4\pi \; [\pi^2 - (\Delta\omega t)^2]} \; (e^{-i\omega_c t} + e^{i\omega_t t} - e^{i\omega_t t} - e^{i\omega_c t}$$

$$= \frac{-i\Delta\omega \; (\Delta\omega t)}{4\pi \; [\pi^2 - (\Delta\omega t)^2]} \; (e^{i\omega_t t} - e^{i\omega_t t} + e^{i\omega_c t} - e^{-i\omega_c t})$$

$$= \frac{-i \; (\Delta\omega)^2 t}{4\pi [\pi^2 - (\Delta\omega t)^2]} \quad (2i \sin \omega_t t + 2i \sin \omega_c t)$$

$$= \frac{(\Delta\omega)^2 t}{2\pi \; [\pi^2 - (\Delta\omega t)^2]} \quad (\sin \omega_t t + \sin \omega_c t) = A_1^* + C_1^*$$

The total answer is

$$A_1^* + C_1^* + B + A_2^* + C_2^* = \frac{(\Delta\omega)^2 t}{2\pi [\pi^2 - (\Delta\omega t)^2]}$$

$$(\sin \omega_t t + \sin \omega_c t) + \frac{\sin \omega_c t}{\pi t} + \frac{1}{2\pi t} \quad (\sin \omega_t t - \sin \omega_c t)$$

$$= \frac{(\Delta\omega)^2 t}{2\pi [\pi^2 - (\Delta\omega t)^2]} \quad (\sin \omega_t t + \sin \omega_c t) + \frac{1}{2\pi t} \quad (\sin \omega_t t + \sin \omega_c t)$$

$$h(t) = \frac{1}{2\pi t} \quad (\sin \omega_t t + \sin \omega_c t) \left[(1 + \frac{(\omega_t - \omega_c)^2 t^2}{(\pi^2 - (\omega_t - \omega_c)^2 t^2)}) \right]$$

$$= \frac{\pi}{2t}\left[\frac{\sin\omega_t t + \sin\omega_c t}{\pi^2 - (\omega_t - \omega_c)^2 t^2}\right]. \tag{2}$$

SECTION IV. THE RESULTING WEIGHT FUNCTION

The weight function just derived from the inverse Fourier transformation is defined in the time domain to be:

$$h(t) = \frac{\pi}{2t}\left[\frac{\sin\omega_t t + \sin\omega_c t}{\pi^2 - (\omega_t - \omega_c)^2 t^2}\right] \tag{2}$$

This is the basic function from which numerical smoothing weights are derived. To obtain weights from this function the following constants must be defined:

1. The number of weights desired. This number must be odd.

2. The sampling interval of the data (Δt).

3. The cutoff and termination frequencies (ω_c and ω_t).

To obtain weights evaluate $h(t)$ at distinct points, h_n, where $h_n = \Delta t\, h(t_n)$; $t_n = n\,\Delta t$; $n = 0, \pm 1, \ldots, \pm N$. When the weights, h_n, have been derived, apply the constraint $\sum_{n=-N}^{N} h_n = 1$.

The weights are now ready to be applied to the data.

A direct Fourier transformation can be applied to the numerical weights to determine the actual filtering effect that application would accomplish. A sum must be substituted for an integral in the Fourier equation. This result can be considered the output filter. The comparison between this filter and the original filter shows the precision of the output filter.

SECTION V. ERROR ANALYSIS AND THE EVALUATION OF SMOOTHING WEIGHTS

The basic equation used in evaluating a set of smoothing weights is the direct Fourier transformation.

$$H(\omega) = \int_{\infty}^{\infty} h(t)\, e^{-i\omega t}\, dt. \tag{3}$$

However, since discrete numerical data are to be used, the equation becomes,

$$\hat{H}(\omega) = \sum_{n=-N}^{N} h(t_n)\, e^{-i\omega t_n}\, \Delta t$$

Since h(t) is evaluated at certain increments, i.e., $t_n = n\,\Delta t$, $h(t_n)$ becomes $h(n\,\Delta t)$, which, with constraints applied becomes h_n (the smoothing weights). Therefore, for any set of (2N + 1) smoothing weights, h_n, where n = 0, ±1, ±2, . . . , ±N the filter or the transfer function can be determined by

$$\hat{H}(\omega) = \sum_{-N}^{N} h_n\, e^{-in\Delta t\omega}$$

This equation can be applied to a set of (2N + 1) weights to determine their smoothing characteristics in the frequency domain. The equation can be rewritten

$$\hat{H}(\omega) = \sum_{-N}^{N} h_n\, (\cos n\, \Delta\, t\, \omega - i \sin n\, \Delta t\, \omega)$$

which reduces to

$$\hat{H}(\omega) = \sum_{-N}^{N} h_n \cos n\, \Delta t\, \omega.$$

In the case of the filter design and weight determination as in this paper where $h_n = h_{-n}$ we have

$$\hat{H}(\omega) = h_o + 2 \sum_{1}^{N} h_n \cos n\, \Delta t\, \omega \tag{4}$$

Equation 4 can be used to determine the actual functional filter output. This equation can be compared with the original design to determine the error involved. Equation 4 can also be used to evaluate any similiar set of numerical weights where $h_n = h_{-n}$. Some evaluations of weight sets are shown in the appendix.

SECTION VI. LIMITATIONS AND FACTORS THAT INFLUENCE ACCURACY

The data sampling frequency, f_s, must be at least twice the highest frequency that is to be considered for filtering purposes. The frequency, f_t, is known as the termination frequency.

Also of special importance is the cutoff frequency, f_c. This is the highest frequency in the basic low-pass filter that has unit gain.

Between the cutoff frequency, f_c, and the termination frequency, f_t, is the roll-off interval where the gain decreases from 1 to 0. The roll-off should occur at frequencies that are not predominant in the data. If there are frequencies occurring in the data at the roll-off interval but have no importance, then a preliminary notch filter can be used to eliminate these frequencies from the data before application of the desired filter.

SECTION VII. CONCLUSION

The results that determine and analyze numerical smoothing weights are so simple that they are likely to be overlooked while reading the text. Numerical smoothing weights are determined by equation 2. Numerical smoothing weights are analyzed by equation 4.

More details of using these equations in digital computer applications are given in the Appendix. Results of applying these equations are also given in the Appendix. Results given are not intended to show the best results of applications but are given because they were available at the time of publication. Rules are given on how to obtain more accuracy and precision.

APPENDIX A-1

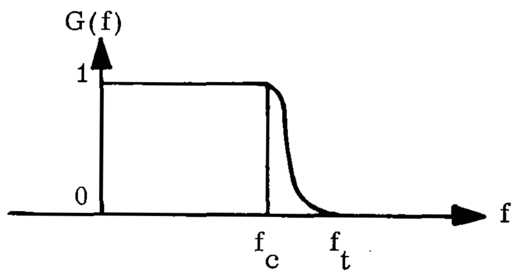
SECTION I. DIGITAL COMPUTER APPLICATIONS

The first equation to be programmed for the digital computer is the weight equation in the time domain, that is;

$$h(t) = \frac{\pi}{2t} \left[\frac{\sin \omega_t t + \sin \omega_c t}{\pi^2 - (\omega_t - \omega_c)^2 t^2} \right] \tag{2}$$

$\omega = 2\pi f,$

where f is the frequency occurring in the data in the time domain. Then $\omega_c = 2\pi f_c$ and $\omega_t = 2\pi f_t$. This weight equation when applied to the data has the effect of a low-pass filter with a cutoff frequency of f_c and a termination frequency of f_t.



f = frequency

G(f) = Gain function of f

$G(f) = H(\omega) = H(2\pi f).$

Plots are usually made with f the independent variable and G(f) the dependent variable.

From equation 2 a set of numerical weights must be determined.

Determine this set of weights as follows:

1. Define f_c the cutoff frequency.
2. Define f_t the termination frequency.
3. Δ t is the time interval between adjacent numerical data values.
4. Define (2 N+1) as the number of weights to be used (where N is an integer).

5. To keep maximum error of the frequency gain in the interval $(0 \leq f \leq f_c)$; at approximately 1 per cent, set $N(\Delta t)\ (f_t - f_c) \geq 2$; at approximately 1/2 per cent, set $N(\Delta t)\ (f_t - f_c) \geq 3$. Δt and f_c are usually fixed; however, f_t and N can vary such that in many cases the conditions above can be met. [The statements above were determined by a series of applications.]

Example 1. $\Delta t = 0.1$, $f_c = 3$ and approximately 1 per cent maximum error is desired, set $f_t = 4$ and we have $N(0.1)\ (4-3) \geq 2$ or $\frac{N}{10} \geq 2$. Hence set $N \geq 20$.

Example 2. $\Delta t = 0.01$, $f_c = 10$, $f_t = 10.5$ and maximum error of approximately 1/2 per cent is desired, so that $N(0.01)\ (0.5) \geq 3$ or $\frac{N}{200} \geq 3$. Hence, set $N \geq 600$.

Find (2N +1) weights $[h_{-n}, h_{-n+1}, \ldots, h_o, \ldots, h_{n-1}, h_n]$ by evaluating equation 2 as follows:

Let

$$t = n\,\Delta\,t,$$

then

$$h(n\,\Delta\,t) = \frac{\pi}{2n\,\Delta t}\left[\frac{\sin \omega_t\, n\,\Delta t + \sin \omega_c\, n\,\Delta t}{\pi^2 - (\omega_t - \omega_c)^2\,(n\,\Delta t)^2}\right]$$

Evaluate for n = 1, 2, 3, . . . , N keeping in mind that $h(n\,\Delta t) = h(-n\,\Delta t)$ thus obtaining 2 N values and $h(o) = f_c + f_t$ which gives the central value of the (2N+1) values.

Now the value must be normalized, that is each $h(n\,\Delta t)$ and $h(o)$ must be divided by $\sum_{n=-N}^{N} h(n\,\Delta t)$ which is the sum of all the (2N+1) values. The latest values just obtained are the weights to be applied to the data and are denoted by $(h_{-n}, h_{-n+1}, \ldots, h_o, \ldots, h_{n-1}, h_n)$. The weights are applied to (2N+1) data points as the scalar or dot product is found in vector analysis, to obtain a smooth data point, which will correspond to the central point of the set of data points to which the weights are applied.

Example: Denote the (2N+1) data points by $d_1, d_2, \ldots, d_{N+1}, \ldots, d_{2N+1}$, then the smoothed data point d'_{N+1} is given by:

$$d'_{N+1} = \sum_{n=-N}^{N} h_n\, d_{(N+n)+1}$$

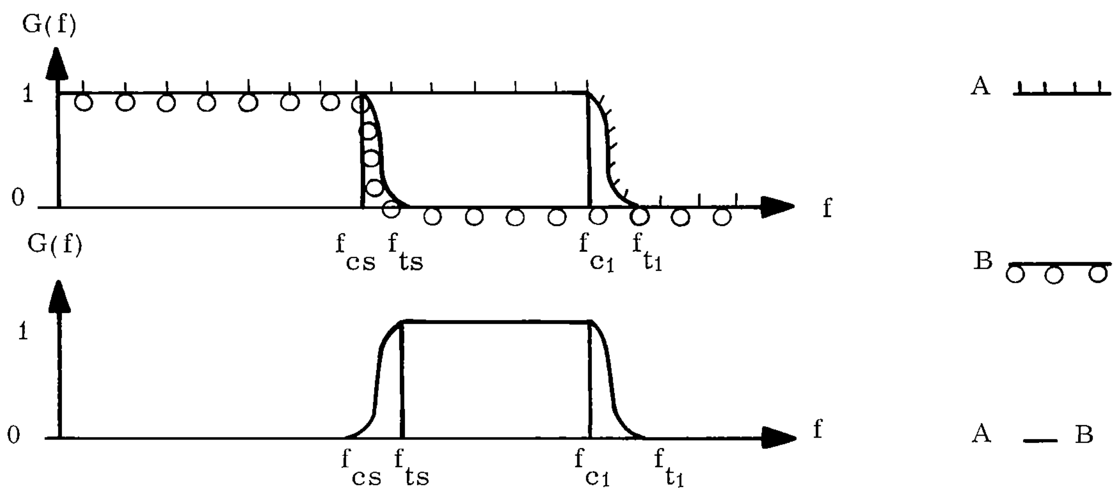
and

$$d'_{N+2} = \sum_{n=-N}^{N} h_n \, d_{(N+n)+2}.$$

Note. In this example the smoothed data points start at the (N+1) point. To have smooth data beginning corresponding to the first data point a special starting procedure must be used. To start the smoothing procedure at the initial point assume data values to the left equal to those to the right, i.e., if raw data starts at d_s, then set $d_{s+k} = d_{s-k}$, $k = 1, \ldots, N$. The same method can be applied to the end point d_e, i.e., $d_{e-k} = d_{e+k}, k = 1, 2, \ldots, N$.

The information above gives details on obtaining a set of weights for digital computer applications. Applying such weights has the same effect on the data as a low-pass filter.

As previously stated, the weights which have the effect of a band-pass filter can be obtained by subtracting corresponding weights of two low-pass filters.



Observe in the frequency domain the difference between two low-pass filters yields a band-pass filter. Since the transformation in the time domain is a linear operation, subtraction of corresponding weights produces the desired results.

Note. For a band-pass filter two cutoff frequencies and two termination frequencies must be given. For example, A above must have a defined cutoff frequency, f_{cl}, and a termination, f_{tl}; likewise B, (f_{cs}, f_{ts}).

The bandwidth is $(f_{cl} - f_{cs})$ or $(f_{tl} - f_{ts})$, and must be selected to give the proper range of frequencies around the center frequency of interest. Too narrow a bandwidth will only give a portion of amplitude of the center frequency; too wide a bandwidth will give amplitudes from undesired adjacent frequencies. Several runs on data have shown that a satisfactory bandwidth is defined by:

$$N \Delta t \text{ (bandwidth)} = 0.5 \tag{6}$$

or

$$\text{bandwidth} = \frac{0.5}{N \Delta t} \tag{6a}$$

The all-pass filter is defined as a filter which will allow all frequencies to pass. The weights in the time domain of such a filter are all zero (0) except the central weight, which is unity, i.e., 0, ..., 0, 1, 0, ..., 0. Thus all data will pass through this filter unchanged. This filter is very useful mathematically to produce other types of filters as shown below.

Weights of a notch filter or band-reject filter are found by subtracting the weights of a band-pass filter from the corresponding all-pass filter weights. That is, change the sign of all the band-pass weights except the central weight which is to be subtracted from unity.

The weights of the all-pass filter minus the corresponding weights of a low pass filter yield the weights of a high-pass filter.

SECTION II. EVALUATION OF A SET OF SMOOTHING WEIGHTS

Refer to Section V in the main body of the text, for review.

$$\hat{H}(\omega) = h_o + 2 \sum_{1}^{N} h_n \cos n \Delta t\, \omega \tag{4}$$

From a set of weights, $\{h_n\}$, N and Δt should be known. Hence the variable in this equation is $\omega = 2\pi f$. Hence f becomes the independent variable.

$$f_s = \frac{1}{\Delta t},$$

hence

$$\frac{f_s}{2} = \frac{1}{2 \Delta t}.$$

$\hat{G}(f) = \hat{H}(2\pi f) = \hat{H}(\omega)$ and $\hat{G}(-f) = \hat{G}(f)$. $\hat{G}(f_s) = h_o + 2 \sum_{1}^{N} h_n \cos 2\pi n$

$$= h_o + 2 \sum_{n=1}^{N} h_n = \hat{G}(0) = \hat{G}(f_s).$$

Consider the function

$$\hat{G}\left(\frac{f_s}{2} - f\right)$$

and

$$\hat{G}\left(\frac{f_s}{2} + f\right)$$

$$\hat{G}\left(\frac{f_s}{2} - f\right) = h_o + 2 \sum_{n=1}^{N} h_n \cos\left[n \Delta t\, 2\pi \left(\frac{f_s}{2} - f\right)\right]$$

$$= h_o + 2 \sum_{n=1}^{N} h_n \left[\cos n \Delta t\, 2\pi \frac{f_s}{2} \cos n \Delta t\, 2\pi f + \sin n \Delta t\, 2\pi \frac{f_s}{2} \sin (n \Delta t\, 2\pi f)\right].$$

Notice, $n \Delta t\, 2 \pi \frac{f_s}{2} = n \pi$ and $\sin n \pi = 0$.

Likewise in $\hat{G}\left(\frac{f_s}{2} + f\right)$, the second term will be zero, and the first term will be the same.

Hence

$$\hat{G}\left(\frac{f_s}{2} - f\right) = \hat{G}\left(\frac{f_s}{2} + f\right).$$

So when $\hat{G}(f)$ is determined in the range $\left(0 \leq f \leq \frac{f_s}{2}\right)$ it is essentially completely determined by the equations above. The results obtained by evaluating $\hat{G}(f)$ are symmetric about a folding frequency, $\frac{f_s}{2}$. This means in some cases amplitudes of undesired higher frequencies will fold back on amplitudes of frequencies in the range of $(0 \leq f \leq \frac{f_s}{2})$. A routine approach to minimizing this problem is to use a higher sampling frequency to reduce the fold-back effect.

SECTION III. ACTUAL DATA RESULTS

The following functions were computed at 0.01 second intervals.

1. $(0.3687) \cos 2\pi t$ — 1 cycle per second (cps) wave
2. $(0.5)(0.3687) \cos [4\pi(t + 0.1)]$ — 2 cps
3. $(0.33)(0.3687) \cos [6\pi(t + 0.1)]$ — 3 cps
4. $(0.25)(0.3687) \cos [8\pi(t + 0.1)]$ — 4 cps
5. $(0.2)(0.3687) \cos [10\pi(t + 0.1)]$ — 5 cps

Then the composite function $f(t) = 1+2+3+4+5$ was computed. The graph of this function is given in Figure 1. The numerical filtering technique was then applied to the composite function in an effort to obtain data of specific cycles of interest. The results of the efforts are given in Figures 2, 3, and 4 and in tabulations following these figures.

Even though application usually extended over several periods of the data involved only one period of data is given in the plot. The parameters involved are stated in the legend of the figures.

The following data verifies that a high-pass filter is the complement of a low-pass filter. $N = 50$, $t = 0.1$, $f_c = 3.99$, $f_t = 4.01$ for low-pass.

Time	A Result Low-Pass	B Result High-Pass Complement	C Sum A + B	Original Composite Data
0.0	+0.3483	-0.1086	+0.2397	+0.2397
0.1	+0.0670	+0.0859	+0.1529	+0.1529
0.2	+0.0756	-0.0577	+0.0179	+0.0179
0.3	-0.0558	+0.0356	-0.0202	-0.0202
0.4	-0.1895	-0.0276	-0.2171	-0.2171
0.5	-0.3099	+0.0350	-0.2749	-0.2749
0.6	-0.3356	-0.0586	-0.3942	-0.3942
0.7	-0.3442	+0.0850	-0.2592	-0.2592
0.8	+0.0941	-0.1091	-0.0150	-0.0150
0.9	+0.6501	+0.1201	+0.7702	+0.7702
1.0	+0.3483	-0.1086	+0.2397	+0.2397

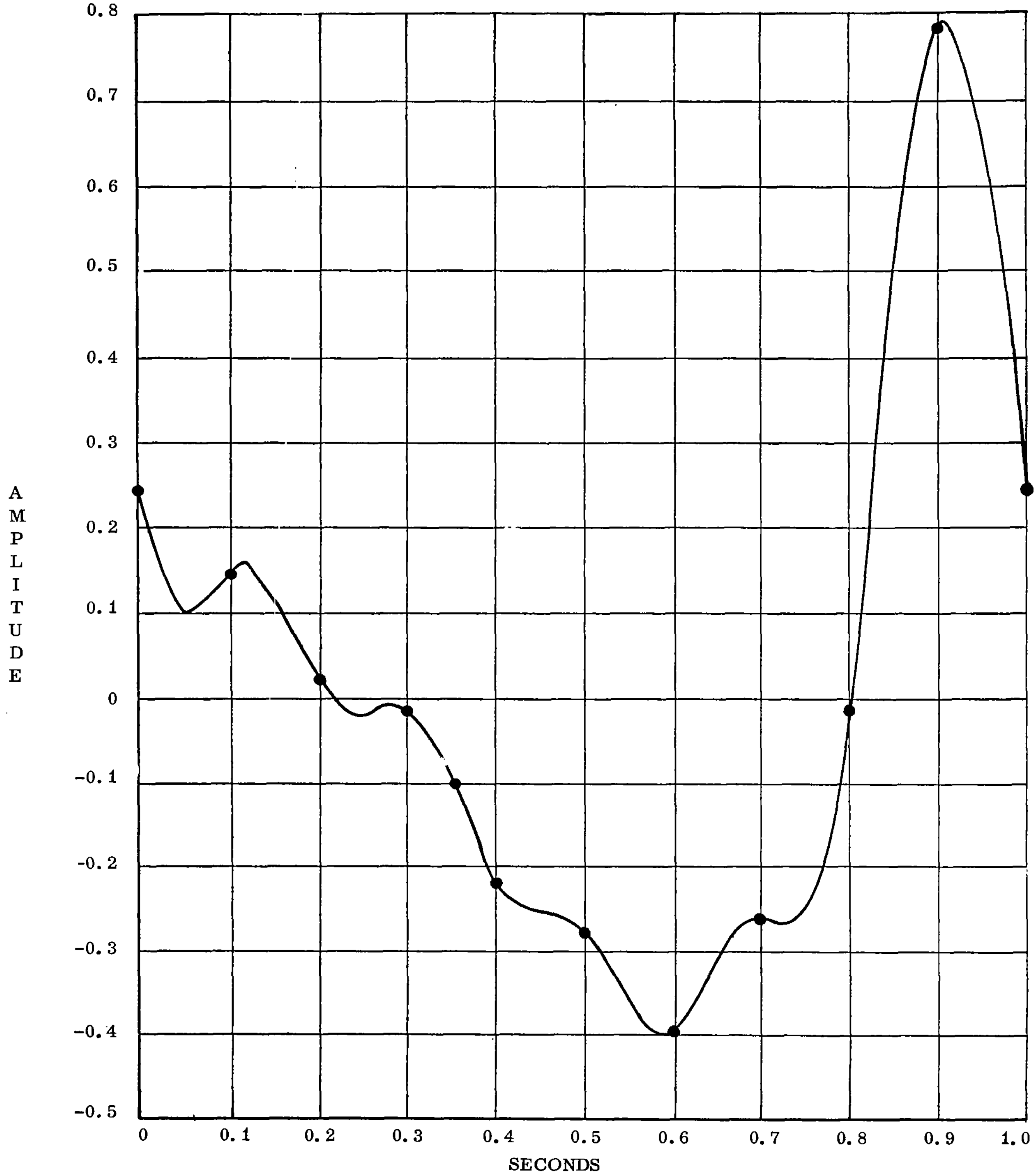


FIGURE 1. ORIGINAL COMPOSITE DATA

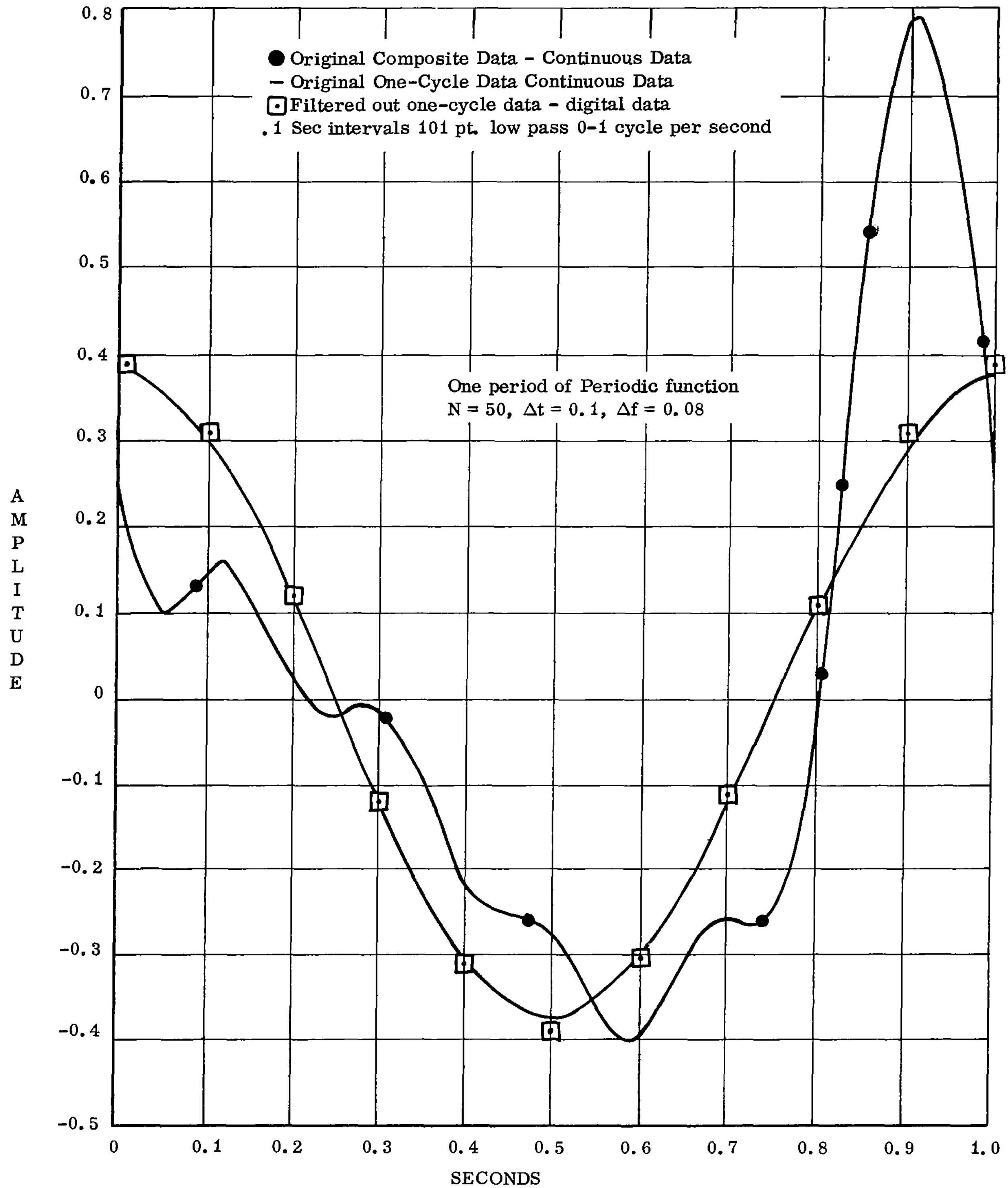


FIGURE 2. COMPOSITE AND FILTERED 0-1 CPS

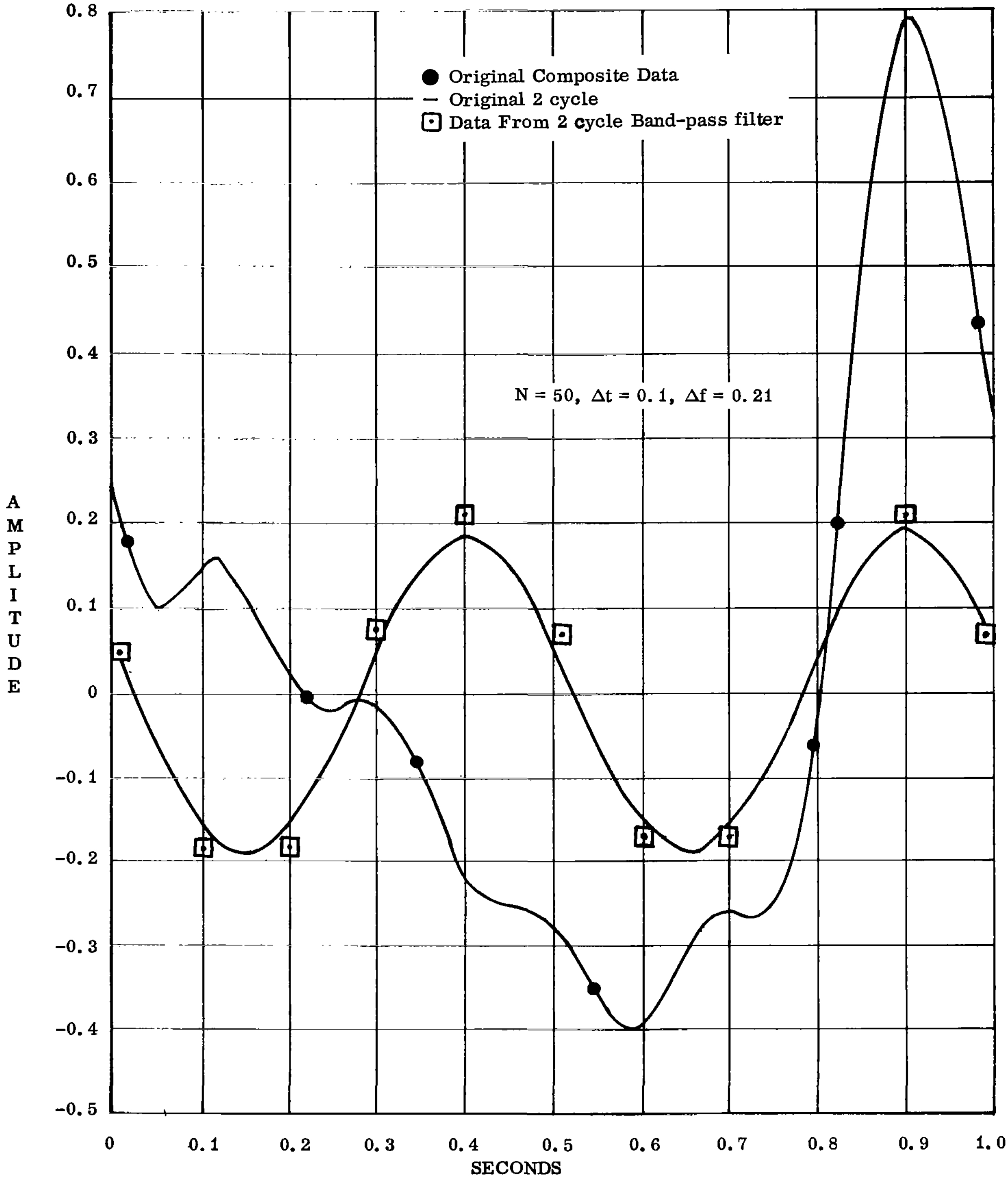


FIGURE 3. COMPOSITE AND 2 CPS BAND-PASS

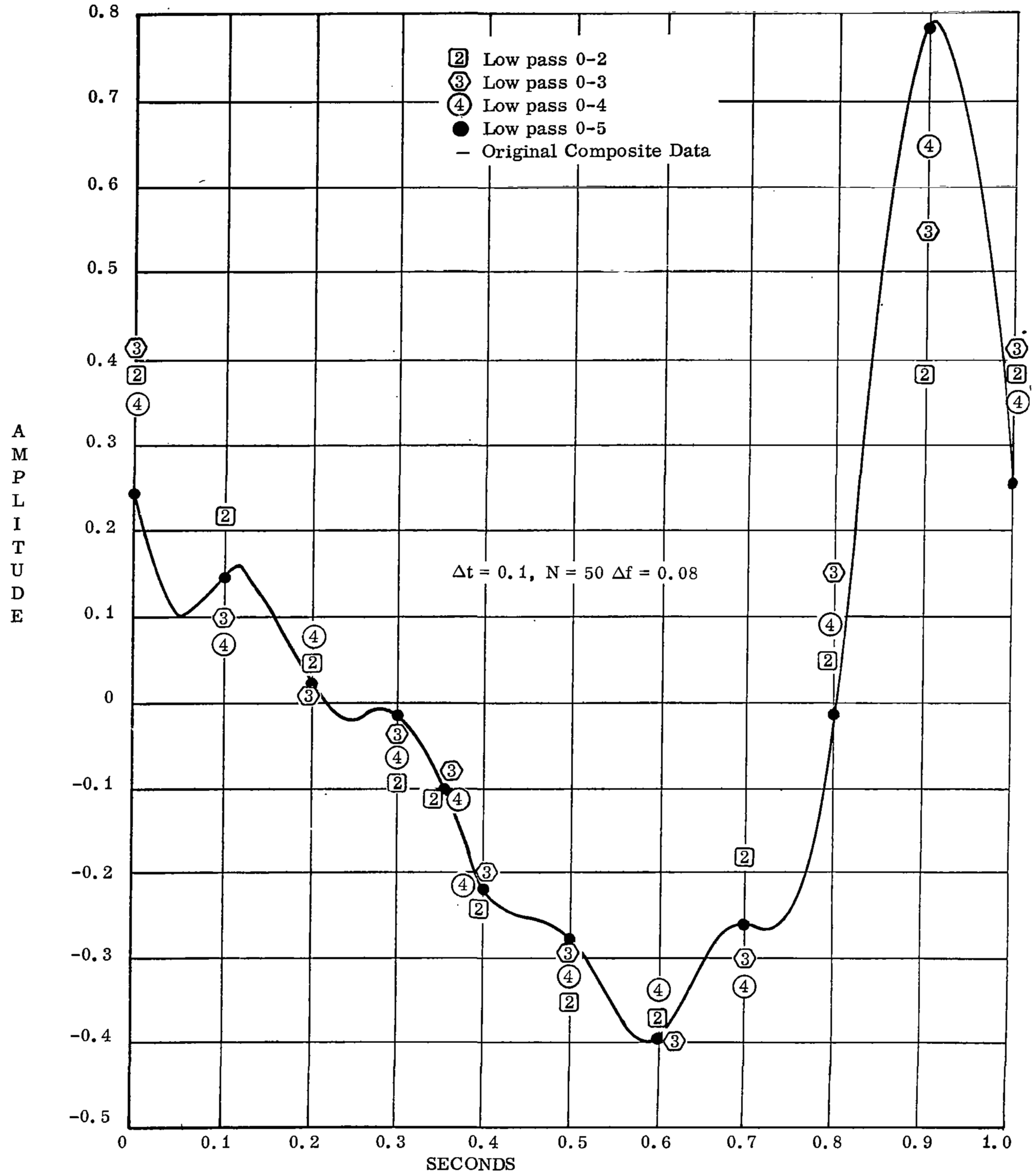


FIGURE 4. 2 CPS, 3 CPS, 4 CPS, AND 5 CPS LOW PASS FILTERS

The following tabulations show the evaluation of band-pass filter weights. Weights were derived from low-pass filters extending slightly beyond the center frequency of interest. The weights were then processed through a correction routine, as described in Reference 1. Corresponding sets of weights were derived, and processed the same, from filters that terminated just prior to the frequency of interest. The heights derived from the over-extended filters minus corresponding weights derived from the under-extended filters yielded the weights of the band-pass filters. The frequency (f) of interest was three cycles per second.

$N = 50,\ \Delta t = 0.1$

Case No.	B. W.	f	$\hat{G}(f)$	Desired $\hat{G}(f)$
1	0.03	3	0.03	1.00
2	0.05	3	0.05	1.00
3	0.07	3	0.07	1.00
4	0.09	3	0.09	1.00

The bandwidths were too small. It is seen when $f = 3$, $\hat{G}(f) << 1$. The data shows the bandwidth (B.W.) should have been equal to 1.

$$N\,(\Delta t)\,(B.\ W.) = 0.5 \tag{6}$$

$$(B.\ W.) = \frac{0.5}{50 \times 0.01} = \frac{0.5}{0.5} = 1$$

The best results were obtained by enforcing equation 6.

A two-cycle band-pass filter was derived using the following parameters:

$N = 50,\ \Delta t = 0.1,\ B.\ W. = 0.21$

For the low-pass filter extending beyond 2 cps, $F_c = 2.07$, $F_t = 2.14$. For the low-pass filter terminating below 2 cps, $f_c = 1.86$, $f_t = 1.93$. The smoothing weights derived were applied to the composite data plotted in Figure 1 at 0.1 second intervals.

The sign of all the band-pass weights was then changed except the central weight, which was subtracted from unity. The resulting weights were then applied to the data (notch or band-reject filter). Results are listed below:

Time	Composite Data	Original 2 Cycle	Band-Pass Result	Notch Result	Band-Pass + Notch
0.0	+0.2397	+0.0569	+0.0649	+0.1748	+0.2397
0.1	+0.1529	-0.1491	-0.1731	+0.3260	+0.1529
0.2	+0.0179	-0.1491	-0.1728	+0.1907	+0.0179
0.3	-0.0202	+0.0569	+0.0661	-0.0863	-0.0202
0.4	-0.2171	+0.1843	+0.2143	-0.4314	-0.2171
0.5	-0.2749	+0.0569	+0.0670	-0.3419	-0.2749
0.6	-0.3942	-0.1491	-0.1713	-0.2229	-0.3942
0.7	-0.2592	-0.1491	-0.1716	-0.0876	-0.2592
0.8	-0.0150	+0.0569	+0.0658	-0.0808	-0.0150
0.9	+0.7702	+0.1843	+0.2109	+0.5593	+0.7702
1.0	+0.2397	+0.0569	+0.0649	+0.1748	+0.2397

Results show that band-pass + notch equals original composite data.

Figure 5 illustrates the actual results of a filter designed with a cutoff frequency of $f_c = 0.65$ and a termination frequency of $f_t = 0.70$. Other parameters were $\Delta t = 0.1$ and $N = 50$. Hence, $N(\Delta t)\ (\Delta f) = 0.25$. It is recommended that $N(\Delta t)\ (\Delta f) > 0.5$ be used.

The plot was determined using $\hat{H}(\omega) = h_0 + 2 \sum_{k=1}^{N} h_k \cos(k\Delta t\omega)$. The maximum error in the interval $0 \le f \le f_c$ occurs near f_c, and in the interval, $f_t \le f \le \frac{f_s}{2}$, the maximum error occurs near f_t.

If the plot had been extended to a frequency = 10, $\hat{H}(10)$ would equal $\hat{H}(0)$. The sampling frequency must be greater than twice the highest frequency to be considered.

Example: A test was run in which the valid data was known to be in the low frequency range of 0-3 cps. The numerical data were sampled at 10 samples per second. A termination frequency of 4 cps seemed reasonable as $\frac{10}{2} > 4$. However, a closer look at the data showed an interference frequency of 10 cps. Such a sampling rate would allow 10 cps to pass with gain of 1. Hence a new sampling rate must be chosen at greater than 20 samples/sec. It is recommended that such data be sampled at 100 samples per second.

Figure 6 illustrates the transformation of Dederick's smoothing coefficients in the time domain to the equivalent filter in the frequency domain. The figure shows the plots of 3 sets of smoothing weights, each consisting of 21 points.

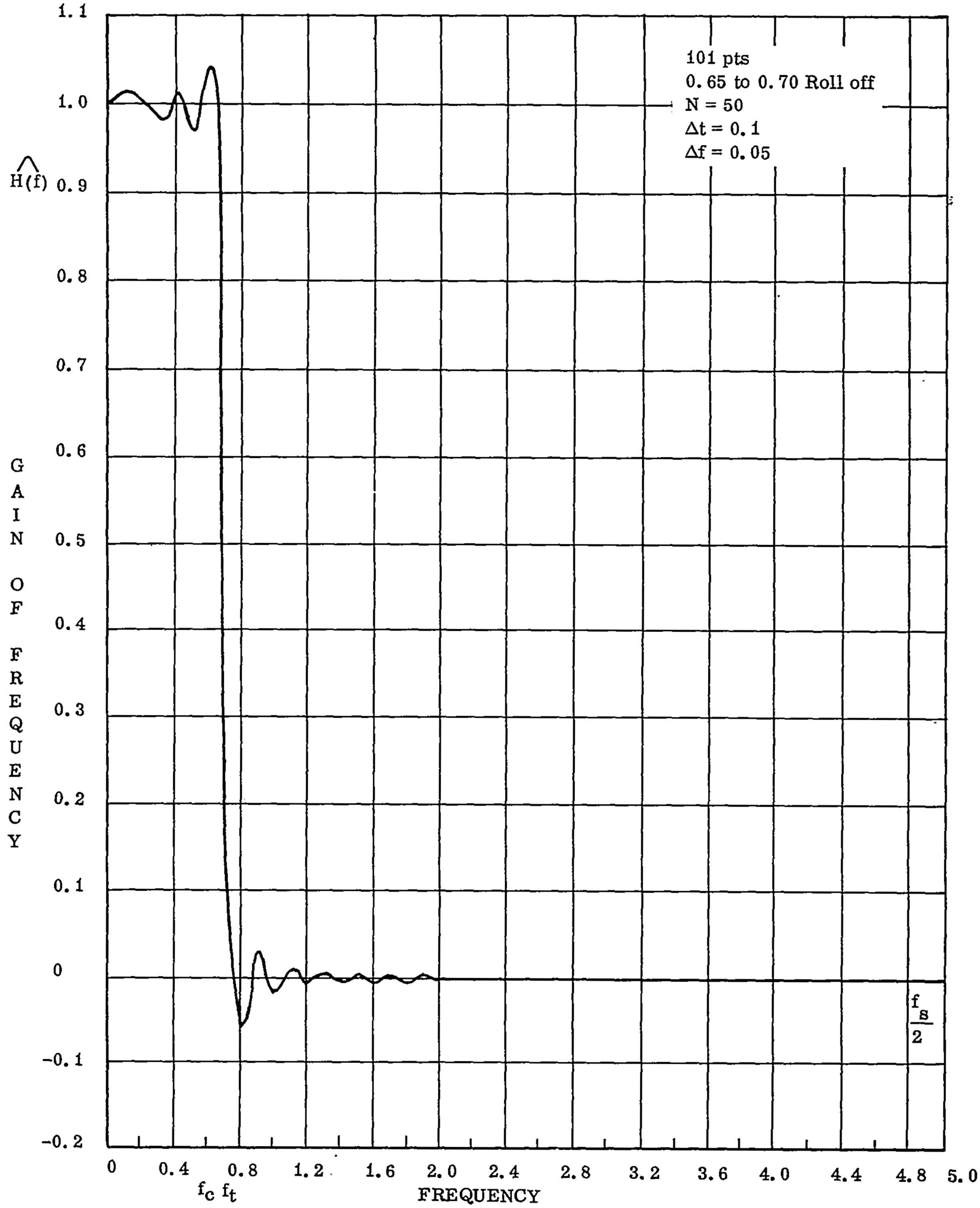


FIGURE 5. LOW PASS FILTER 0-0.65

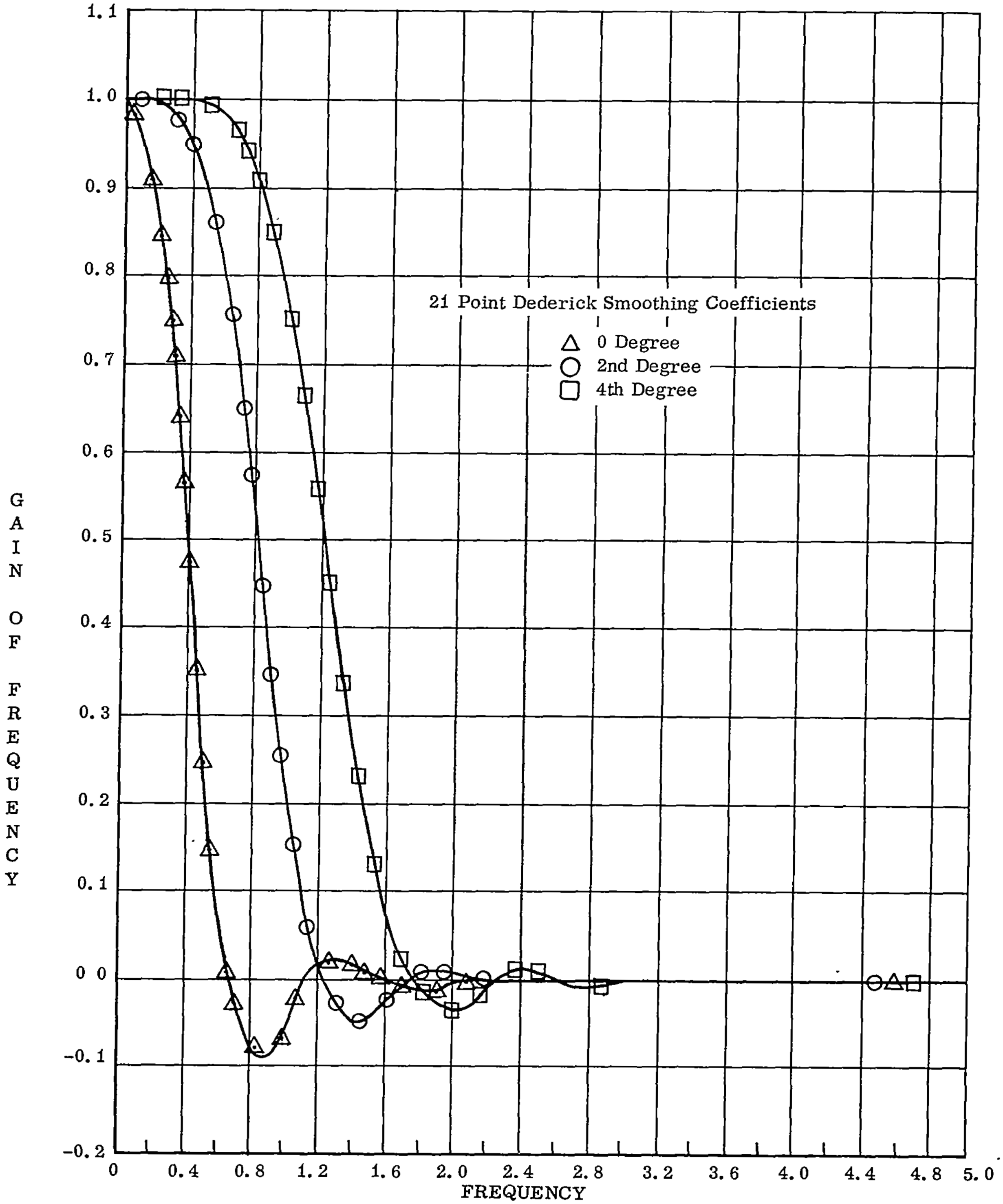


FIGURE 6. 21 POINT DEDERICK SMOOTHING COEFFICIENTS

Equation 4 is used here to evaluate a set of weights that are determined through the filter technique. The transfer functions were based on weights to be applied to data at 0.1-second intervals, i.e., $\Delta t = 0.1$.

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3. Fleck, J. T. and Fryer, W. D., "An Exploration of Numerical Filtering Techniques," Cornell Aeronautical Laboratory, Inc. Buffalo, N. Y., Report No. XA-869-P-1.

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